

Bachelor Seminar: Business Cycles, Theory and Empirical Applications

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Abstract

This seminar paper covers the macroeconomic development of Japan and the compatibility of two different RBC models with the recovered statistics.

The first section covers the computation of relevant statistics for Japan and the interpretation of these results. In order to do so, data inputs for all relevant variables had to be found. For most of the following computation, the open-source statistical software gretl was used. The data has been detrended and relevant macroeconomic statistics were recovered. Additionally, TFP was computed as Solow residual on basis of two different production functions.

The second section covers the calibration of a simple stochastic RBC model. Later, the statistics generated by the model are compared to those recovered in the first part. It ends with a conclusion about the validity of the used model.

The third section covers the solution of an RBC model with endogenous labor by hand and in Dynare. Later, the statistics generated by simulating the model are compared to those recovered in the first part. It ends with a conclusion about the validity of the used model, how it improved and what can still be changed in order to improve its performance.

Contents

1	Data Work	1
1.1	Recovering Relevant Macroeconomic Data	1
1.1.1	Gross Domestic Product	1
1.1.2	Private Consumption	1
1.1.3	Private Investment	1
1.1.4	Labor Supply	2
1.1.5	Capital Stock	2
1.2	Detrending the Data	3
1.2.1	Introducing the Hodrick-Prescott-Filter	3
1.2.2	Detrended Data	3
1.3	Calculating the Basic Statistics	6
1.4	Estimating Total Factor Productivity	9
1.4.1	First Approach	9
1.4.2	Second Approach	11
2	The Simple Stochastic RBC Model	12
2.1	Calibrating the Model	12
2.1.1	First Procedure	12
2.1.2	Second Procedure	13
2.2	Simulating the Model in Excel	13
2.3	Evaluating the Model's Performance	14
2.3.1	How to Handle the Data	14
2.3.2	Standard Deviation	14
2.3.3	Correlation with GDP	15
2.3.4	Autocorrelation	15
2.3.5	Lead and Lag Analysis	17
2.4	Conclusion	20
3	The RBC Model with Endogenous Labor	21
3.1	Solving the Model by Hand	21
3.1.1	Combining the Constraints	21
3.1.2	Setting up the Lagrangian and Deriving the FOCs	21
3.1.3	Expressing FOCs in Terms that do not grow in Steady State	22
3.1.4	Characterizing the Nonstochastic Steady State	23
3.1.5	Linearizing Around the Steady State	23
3.1.6	Campbell's Method to Solve the System	26
3.1.7	Evaluating the Solution Derived by Hand	28
3.2	Solving the Model in Dynare	28
3.3	Evaluating the Model's Performance	30
3.3.1	Standard Deviation and Correlation with GDP	31
3.3.2	Autocorrelation	31
3.3.3	Lead and Lag Analysis	33
3.4	Conclusion	34

1 Data Work

1.1 Recovering Relevant Macroeconomic Data

Since Japan is a long-standing member of the Organisation for Economic Co-operation and Development (OECD), it was straightforward to search for data in their statistical database. Hence, most of the data was extracted from the OECD Economic Outlook No. 98 as at November 2015.

Quarterly data from Q1 1960 to Q4 2017 (estimates) was used and all variables (except for N_t) were converted in Yen 2005 units.

1.1.1 Gross Domestic Product

Data on Japan's Gross Domestic Product (GDP) as proxy for what is Y_t was recovered from the OECD's Economic Outlook 98 as "Gross domestic product, volume, market prices".

The data was provided in quarterly frequency, fitted to the official annual figures. The procedure used by the OECD's statistical office to recover data in higher frequency follows the method presented by Chow and Lin¹.

After being interpolated into higher frequency, seasonal adjustment was conducted by the OECD. To do so, the United States Bureau of the Census' X 12 ARIMA (Autoregressive Integrated Moving Average) seasonal adjustment program was used.

The data was presented in 2005 Yen units.

1.1.2 Private Consumption

Data on Japan's private consumption as proxy for what is C_t was recovered from the OECD's Economic Outlook 98 as "Private final consumption expenditure, volume".

Just as the data on GDP, private final consumption expenditure was provided in quarterly frequency, seasonally adjusted and in 2005 Yen.

1.1.3 Private Investment

Data on Japan's private investment as proxy for what is I_t was recovered from the OECD's Economic Outlook 98 as "Gross fixed capital formation, total, volume".

Gross fixed capital formation (GFCF) "is measured by the total value of a producer's

¹Chow, G. C. and A.-I. Lin (1971) "Best linear unbiased interpolation, distribution, and extrapolation of time series by related series", The Review of Economics and Statistics 53(4): 372-375.

acquisitions, less disposals, of fixed assets during the accounting period”². It therefore is not a measure of total investment since i.e. investment in human capital or financial assets is neglected.

Nevertheless, this data set was chosen as proxy for I_t since instances like the IMF encourage researchers to use GFCF as proxy for private investment³.

Just like the previous data series extracted from the OECD’s Economic Outlook 98, private investment was already in quarterly frequency, seasonally adjusted and in 2005 Yen.

1.1.4 Labor Supply

Data on Japan’s labor supply as proxy for what is N_t was recovered from the OECD’s Economic Outlook 98 as ”Total employment”.

The data was in quarterly frequency and in person units. Since seasonal adjustment was missing in this dataset, it had to be conducted afterwards. In order to compute data consistent with the previously recovered, the X 12 ARIMA process had to be applied. This was done by using the gretl add-on package ”x12a”, which installs a function that allows for bespoke analysis.

1.1.5 Capital Stock

Data on Japan’s capital stock as proxy for what is K_t was recovered from the European Commission Annual Macro-Economic Database as ”Net capital stock at 2010 prices: total economy - Mrd JPY - Japan”.

Since the values are displayed in 2010 Yen, they had to be converted into 2005 Yen before being used for the analysis. This has been done by using a deflator:

$$K_{t,2005Yen} = K_{t,2010Yen} \frac{Deflator_{2005Yen}}{Deflator_{2010Yen}}$$

Since the data was provided in annual frequency, a timeseries disaggregation had to be conducted as well. This was done by using gretl’s built in GUI function ”Expand data”. This method enables to neglect the seasonal adjustment, since no quarterly data was used to interpolate annual data (as with Chow Lin) but the annual data itself. This means that no quarterly noise was introduced, which might lead to lower than actual standard deviation.

²Glossary of statistical terms, OECD, <https://stats.oecd.org/glossary/detail.asp?ID=1171> , reviewed at 9.6.2016

³<http://datahelp.imf.org/knowledgebase/articles/536206-where-do-i-find-data-on-private-investment> , reviewed at 9.6.2016

1.2 Detrending the Data

1.2.1 Introducing the Hodrick-Prescott-Filter

In order to detrend the data, natural logarithms of the recovered data had to be taken.

This enabled to apply the Hodrick-Prescott-Filter, which minimizes the magnitude⁴:

$$\sum_{t=1}^T (y_t - g_t)^2 + \lambda \sum_{t=2}^{T-1} [(g_{t+1} - g_t) - (g_t - g_{t-1})]^2$$

y_t = log of the data in period t ; g_t = log of the growth component in period t

When working with data in quarterly frequency, the value for λ proposed by Hodrick and Prescott⁵ is $\lambda = 1600$.

1.2.2 Detrended Data

Applying the Hodrick-Prescott-Filter to the recovered data decomposed the variable into its trend and cyclical component. The results are being displayed along with information about highly relevant historical events that help to explain some remarkable outliers or uniqueness in development.

The impact of the oil crisis in 1973 that resulted from the Fourth Middle Eastern War can be seen in all cyclical components. While the import ratio of oil for OECD countries averaged at 67%, Japan imported 99.7% of its oil used either for consumption or for production of i.e. plastics⁶. This made it highly dependent on the market prices which approximately four-folded from 3\$ to nearly 12\$.

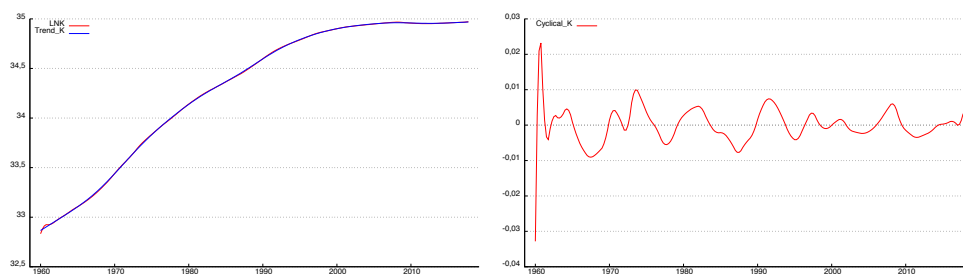


Figure 1: The logarithmized Capital Stock and its Trend

⁴Peter Birch Sorensen and Hans Jorgen Whitta-Jacobsen, *Introducing Advanced Macro-economics: Growth & Business Cycles*, first edition, McGraw-Hill 2005, pages 403-405

⁵Hodrick, Robert J, and Edward C. Prescott. "Postwar U.S. Business Cycles: An Empirical Investigation." *Journal of Money, Credit, and Banking*. Vol. 29, No. 1, February 1997, pp. 1-16., available at: <http://www.jstor.org/stable/2953682>

⁶Kenichi Ohno, *The Path Traveled by Japan as a Developing Country: Economic Growth from Edo to Heisei*, Yuhikaku Publishing Co. Ltd., Tokyo, 2005, page 186.

The impact of the financial crisis in 2008 on Japan's private investment can be seen very clearly as well even though its relatively resilient financial system initially limited the direct impact. But since over 90% of Japan's exports consisted of highly income-elastic industrial supplies, capital goods, and consumer durables, it has been very responsive to output shocks in the advanced markets of the United States and Western Europe. One example is constituted by the exports to Emerging Asia as Japan's largest export market. Japan mostly exported intermediate goods used in the production of final goods. As demand for these goods in the Western World began to drop, demand for the intermediate Japanese goods declined as well⁷.

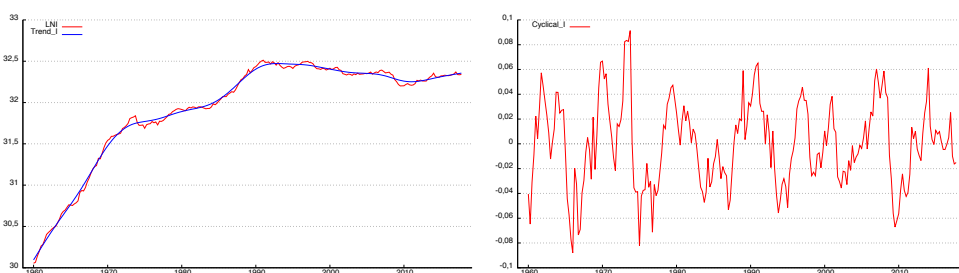


Figure 2: The logarithmized Private Investment and its Trend

What is remarkable to notice is that the Japanese economic post-war system was built on long-term relationships. Features such as lifetime employment help to explain the very small deviations of total employment over time.

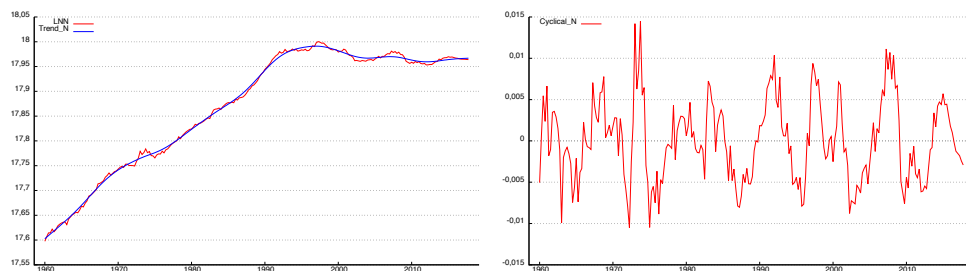


Figure 3: The logarithmized Total Employment and its Trend

It is argued that this system started to become obsolete by the 1970s. According to the view of some economists, Japan should have shifted its economic system to a

⁷Kawai, M., and S. Takagi. 2009. Why was Japan Hit So Hard by the Global Financial Crisis? ADBI Working Paper 153. Tokyo: Asian Development Bank Institute. Available: <http://www.adbi.org/working-paper/2009/10/05/3343.japan.gfc/>

more market-based one, in order to adapt to its industrial society that emerged over the previous 20 years⁸.

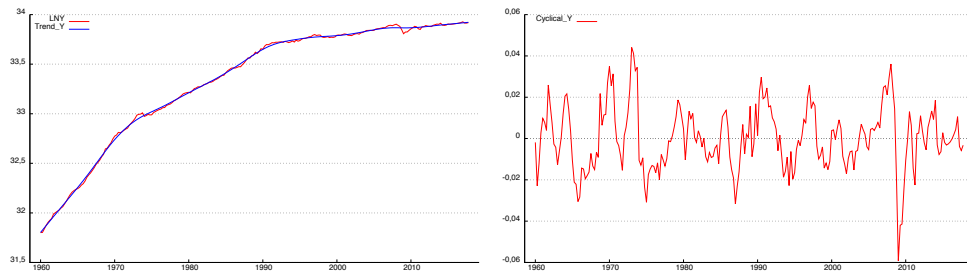


Figure 4: The logarithmized GDP and its Trend

The logarithmized data and growth component show approximately three "states of growth". It starts with the catching-up process after World War II and the Korean War until 1970, which is characterized by an extraordinary high growth rate of approximately 10%. After catching up with other developed economies, the growth rate decreased in 1971 to an average of about 4%. A slowdown in growth is common for maturing economies. Additionally, inflation accelerated in all industrial countries around that time⁹. The well known "lost decade" started around 1991 and came with an extremely low growth rate of GDP at near zero as seen in the graph.

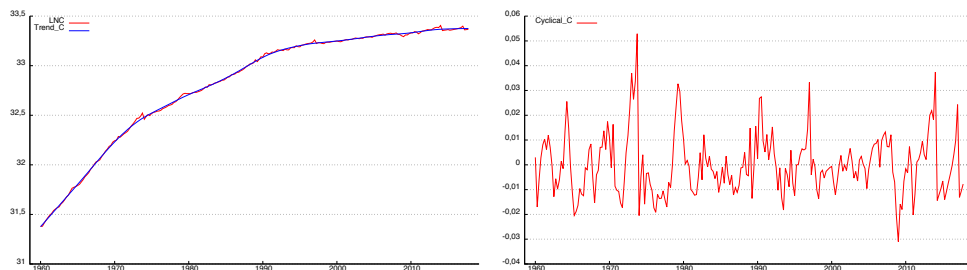


Figure 5: The logarithmized Private Consumption and its Trend

Private consumption accounts for about 60 percent of Japan's GDP, and therefore holds the key to the country's economic health. The lack of consumer confidence is crippling economic activity. The bearish sentiment is said to be one of the major factors

⁸Kenichi Ohno, *The Path Traveled by Japan as a Developing Country: Economic Growth from Edo to Heisei*, Yuhikaku Publishing Co. Ltd., Tokyo, 2005. page 190

⁹Kenichi Ohno, *The Path Traveled by Japan as a Developing Country: Economic Growth from Edo to Heisei*, Yuhikaku Publishing Co. Ltd., Tokyo, 2005, pages 162 and 184

that prevent the economy from being actively simulated. It is now the governments responsibility to restore public confidence¹⁰.

Plotting all cyclical components of the variables together creates the following diagram, allowing for initial guesses about volatility, leads and lags.

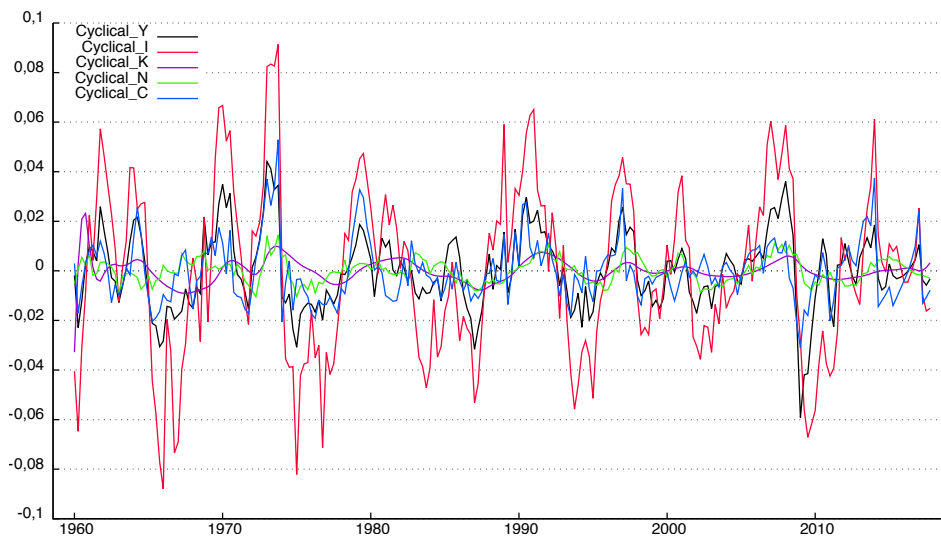


Figure 6: Cyclical Components of all Variables

The following statistical analyses are all covering the data recovered for the cyclical components. This holds true for the comparison against model generated data as well.

1.3 Calculating the Basic Statistics

At first, the standard deviation was analyzed. Calculating the standard deviation follows:

$$\sigma_x = \sqrt{\frac{1}{N} \sum_{i=1}^N (x_i - \mu_x)^2}$$

In order to provide a convenient way to compare the standard deviation, it was converted to relative standard deviation to GDP. As can be seen in the chart above, investment clearly has the highest standard deviation and is the only variable that is more volatile than GDP.

¹⁰Pradyumna Prasad Karan, Japan in the twenty-first century: Environment, Economy, and Society, The University Press of Kentucky 2005, page 318

Afterwards, the correlation of the variables with GDP were examined since it is used as main indicator of economic activity. The correlation of given variables x_t, c_t is defined as:

$$\rho_{x_t, c_t} = \frac{\sigma_{x_t c_t}}{\sigma_{x_t} \sigma_{c_t}} = \frac{\sum_{t=1}^T (x_t - \mu_x)(c_t - \mu_c)}{\sqrt{\sum_{t=1}^T (x_t - \mu_x)^2} \sqrt{\sum_{t=1}^T (c_t - \mu_c)^2}}$$

It can be seen that consumption has a lower standard deviation than GDP. This can mostly be attributed to the fact, that consumption decisions are mostly based on averages of previous income changes¹¹. Capital and labor both have even lower standard deviations.

	GDP	Consumption	Capital	Investment	Labor
Std	0.0153	0.0123	0.0049	0.0343	0.0048
Rel. Std to GDP	1.0000	0.8002	0.3222	2.2385	0.3165
Corr. with GDP	1.0000	0.7454	0.3086	0.8601	0.4585

Table 1: Standard Deviation, Rel. Std. and Correlation with GDP

It is straightforward to see that all variables move approximately in the same direction, since all correlations are positive. N_t and K_t have lower correlations with GDP than the residual variables.

Autocorrelation is the correlation of a variable with its own lagged value. It is used to describe the persistence of a variable.

Lag	GDP	Consumption	Capital	Investment	Labor
t+1	0.7789	0.5260	0.7196	0.8245	0.7753
t+2	0.5514	0.3981	0.4721	0.6465	0.6369
t+3	0.3275	0.2780	0.3306	0.4713	0.4522
t+4	0.0967	-0.0167	0.3065	0.2897	0.2843
t+5	-0.0610	-0.1289	0.2829	0.1364	0.1470

Table 2: Observed Autocorrelation

From the table, it can be seen that investment has the highest persistence over three quarters. Afterwards, capital takes the spot.

Whether a variable is considered as leading or lagging is determined using crosscorrelations with current GDP over time. The correlation of current values is defined as ρ_{x_t, c_t} . If the lagging correlation ρ_{x_{t+a}, c_t} with a lag of the size a is greater than the current correlation, the variable x will be considered a lagging variable. This means, the future value of this variable has a greater correlation with current GDP than the current value of the variable has.

¹¹Campbell, John Y., and Angus Deaton, "Is Consumption Too Smooth?", Review of Economic Studies 56 (July 1989), 357-373

The same holds for leading variables with the leading correlation ρ_{x_{t-a},c_t} . In this case, a previous value of the variable has a greater correlation with current GDP than its current value. Leading indicators are thus useful in the attempt to estimate or predict future GDP.

Lag	Consumption	Capital	Investment	Labor
t-5	-0.0029	-0.2111	0.0317	-0.1744
t-4	0.1415	-0.1443	0.1632	-0.1116
t-3	0.3582	-0.0479	0.3420	0.0282
t-2	0.4620	0.0804	0.5083	0.1853
t-1	0.6063	0.2437	0.6977	0.3476
t	0.7457	0.4071	0.8607	0.4708
t+1	0.4863	0.5338	0.7811	0.5227
t+2	0.3224	0.6087	0.6478	0.5329
t+3	0.1709	0.6324	0.4720	0.4690
t+4	-0.0222	0.6088	0.2766	0.3569
t+5	-0.1102	0.5545	0.0889	0.1959

Table 3: Observed Crosscorrelation

The shown data can also be presented in a chart, allowing for easier and faster identification of the leading and lagging variables.

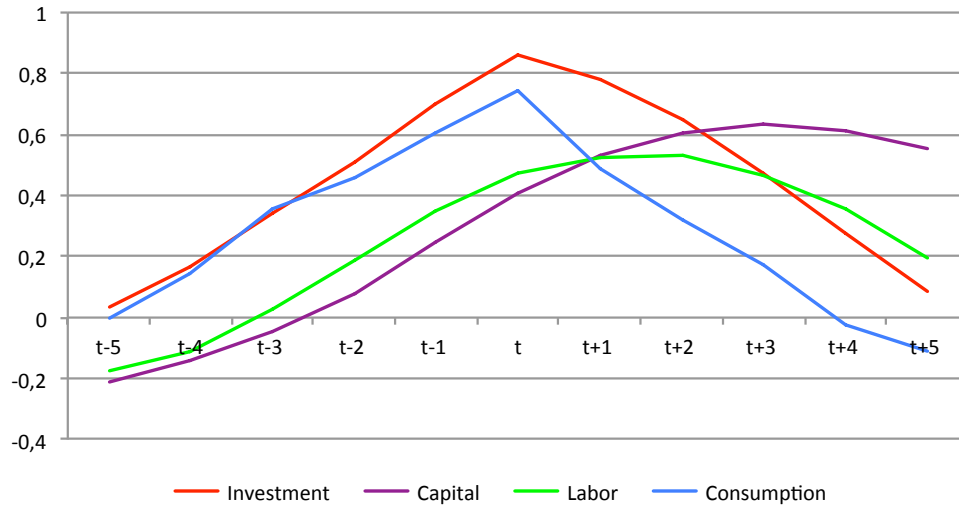


Figure 7: Observed Crosscorrelation

Future capital has a higher correlation with current GDP. This finding supports cur-

rent theory, declaring capital as lagging variable. Capital is a lagging variable partly due to its time-to-build feature¹².

Investment has the highest correlation with GDP in the absence of lags and is thus a coincident variable. This finding does support current macroeconomic theory since investment is widely accepted as a leading variable only if all sorts of investment are being taken into account. Keeping in mind that only GFCF was used as proxy for investment, it is obvious that this is not the case; investment in financial assets (e.g. investment in stocks that typically rises prior to GDP¹³) is missing.

Labor is a lagging variable since its correlation of future values with current GDP is higher than of current values. It has to be remembered that total employment was used as proxy for labor. One possible explanation for why labor is a lagging variable might be that employers can not and do not want to fire their employees as soon as output drops. After investing in an employee, the representative firm would prefer to decrease the hours worked or implement a part-time contract to firing the employee right away in order not to lose the upfront investment in training and the accumulated expertise of this particular employee¹⁴.

Consumption is neither a clearly leading or lagging variable but coincident.

1.4 Estimating Total Factor Productivity

There are two different approaches to recover total factor productivity.

1.4.1 First Approach

The first approach was based on the following production function:

$$Y_t = e^{A_t} K_t$$

Recovering the total factor productivity first required rearrangement of the given equation to:

$$A_t = \ln(Y_t) - \ln(K_t)$$

This formula was applied to the undetrended data. Running a linear regression on A_t over time yielded:

¹²Ralph Winkler, Ulrich Brandt-Pollmann, Ulf Moslener and Johannes Schlöder, On the Transition from Instantaneous to Time-Lagged Capital Accumulation - The Case of Leontief Type Production Functions, Discussion Paper No. 05-30, Centre for European Economic Research

¹³M. Burda and C. Wyplosz, Macroeconomics - A European Text, sixth edition, Oxford University press 2013, page 419

¹⁴Mark Thoma, What Causes Employment to Lag Output in Recoveries?, CBS money watch, available at <http://www.cbsnews.com/news/what-causes-employment-to-lag-output-in-recoveries/>, reviewed on 30.5.2016

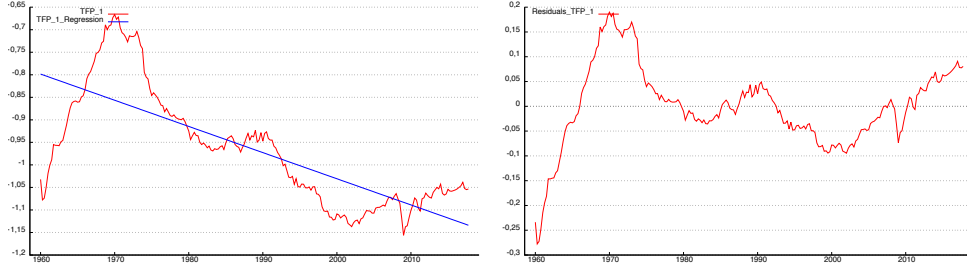


Figure 8: TFP, its Linear Regression Line and the Estimation Error

	Coefficient	Standard Error (SE)	t-quotient	p-value
Constant	-0.797016	0.0107509	-74.14	1.50e-162
time	-0.00145180	8.00045e-05	-18.15	2.85e-46

Table 4: The Regression Parameters

For testing the significance of both variables, either the t-quotient or the p-value can be used. Both measures show significance of both parameters at a 1% significance level. This resulted in the following regression function:

$$\hat{A}_t = -0.797016 + -0.00145180 * t$$

where t represents one quarter.

Since data from Q1:1960 until Q4:2017 was used, T=232 points in time were obtained. The properties of the above regression were:

Average of estimated variables	-0.966151	R^2	0.588769
Sum of quad. res.	1.531904	SE of the regression	0.081612
$F(1, 230)$	329.2968	P-value(F)	2.85e-46

Subtracting values estimated by the regression from actual data yields the detrended TFP. Since detrended TFP was recovered as residual, this operation equals:

$$u_t = A_t - (\beta_0 + \beta_1 * t) \text{ equal to } u_t = A_t - \hat{A}_t$$

Now, a first order autoregressive process had to be applied to the detrended TFP $\hat{u}_t = \rho u_{t-1} + \epsilon$ in order to recover the metrics that will be used in the coming models. The metrics were recovered as:

$$\rho_A = 0.956538 \text{ and } \sigma_\epsilon = 0.010958$$

1.4.2 Second Approach

The alternative approach was based on a production function that included labor and positive but decreasing marginal products. It had to be rearranged to solve for TFP as well.

$$Y_t = e^{A_t} K_t^{0.36} N_t^{0.64}$$

$$A_t = \ln(Y_t) - 0.36\ln(K_t) - 0.64\ln(N_t)$$

Recovering TFP with the alternative approach and running a linear regression on it yielded:

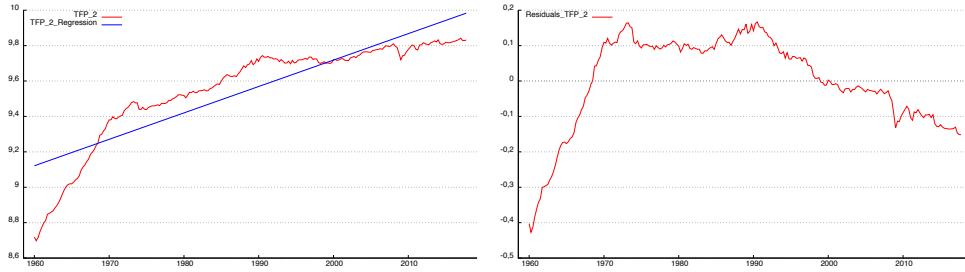


Figure 9: TFP 2, its Linear Regression Line and the Estimation Error

	Coefficient	Standard Error (SE)	t-quotient	p-value
Constant	9.11759	0.0171364	532.1	0.0000
time	0.00373183	0.000127524	29.26	1.71e-79

Table 5: The Regression Parameters

For testing the significance of both variables, one can again use either the t-quotient or the p-value. Both measures show significance of both parameters at a 1% significance level as well. This yielded the following regression function:

$$\hat{A}_t = 9.11759 + 0.00373183 * t$$

The statistical properties of the regression were:

Average of estimated variables	9.552352	R^2	0.788287
Sum of quad. res.	3.892098	SE of the regression	0.130085
$F(1, 230)$	856.3740	P-value(F)	1.71e-79

After running the first order autoregressive process $\hat{u}_t = \rho u_{t-1} + \epsilon$ on the TFP-residuals, which were recovered in the same way as before, the following metrics were received:

$$\rho_A = 0.971296 \text{ and } \sigma_\epsilon = 0.010987$$

2 The Simple Stochastic RBC Model

This section covers the calibration of the simple stochastic RBC model and the evaluation of the simulation's statistics.

The considered model is defined as:

$$\max_{(C_t, I_t, K_t)_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \log C_t$$

s.t.

$$C_t + I_t = \theta \exp(A_t) K_t$$

$$K_{t+1} = (1 - \delta) K_t + I_t$$

$$K_0 \quad \text{given}$$

Its solution was given by

$$K_{t+1} = \beta[(1 - \delta)K_t + \theta \exp(A_t)K_t]$$

2.1 Calibrating the Model

2.1.1 First Procedure

Building upon the solution of the model, the normalization parameter θ had to be calibrated by rearranging the model's solution in steady state.

Being in steady state, the solution becomes

$$K = \beta[(1 - \delta)K + \theta \exp(0)K]$$

since

$$K_t = K_{t+1} = K \quad \text{and} \quad A_t = 0 \quad \forall t$$

Which then can be rearranged to:

$$\frac{1}{\beta} - 1 + \delta = \theta$$

Since quarterly data was used in the first part, $\beta = 0.99$ and $\delta = 0.025$ are the relevant parameters. This allows to solve the equation above:

$$\frac{1}{0.99} - 1 + 0.0225 = \frac{139}{3960} = \theta$$

Now the three ratios $\frac{I}{K}$, $\frac{K}{Y}$ and $\frac{I}{Y}$ that describe the steady state can be calculated:

$$\begin{aligned}
K &= (1 - \delta)K + I & Y &= C + I = \theta \exp(A)K & \frac{I}{Y} &= \frac{I}{Y} * \frac{K}{K} \\
1 &= (1 - \delta) + \frac{I}{K} & \frac{Y}{K} &= \theta \exp(0) & \frac{I}{Y} &= \frac{I}{K} * \frac{K}{Y} \\
\frac{I}{K} &= \delta & \frac{K}{Y} &= \frac{1}{\theta} & \frac{I}{Y} &= \frac{\delta}{\theta} \\
\frac{I}{K} &= 0.025 & \frac{K}{Y} &= \frac{3960}{139} & \frac{I}{Y} &= 0.025 \frac{3960}{139} = \frac{99}{139}
\end{aligned}$$

2.1.2 Second Procedure

This procedure assumes $\frac{K}{Y} = 12$. Since $\frac{K}{Y} = \frac{1}{\theta}$, $\theta = \frac{1}{12}$. Since $\beta = 0.99$ is still assumed, inserting this finding in the equation, used to determine θ in the first procedure, enabled to solve for δ .

$$\begin{aligned}
\frac{1}{\beta} - 1 + \delta &= \theta \\
\delta &= \frac{1}{12} - \frac{1}{0.99} + 1 = \frac{29}{396}
\end{aligned}$$

2.2 Simulating the Model in Excel

In order to create the described model in Excel, a random series of shocks of the size σ_ϵ or $-\sigma_\epsilon$, each with probability 0.5 had to be created.

This was done using the Excel function:

$$=IF(RANDBETWEEN(0;1000000)/1000000<0.5;-B4;B4)$$

Where the cell B4 contained the corresponding σ_ϵ . The parameters recovered by the AR(1) process in subsections 1.4.1 and 1.4.2 have to be used in order to create values for A_t following $A_t = \rho A_{t-1} + \epsilon$.

Since two different approaches to uncover TFP were applied, this has to be done for both pairs of parameters. The other relevant parameters were just calculated by using two different approaches as well. This implies four possible combinations. The model therefore had to be run four times in total.

Thus, in the following analysis, the model with different calibrations will be labeled as follows, indicating which results were used for the simulation:

Name	Method used to recover TFP	Method of calibration
TFP1 M1	The first method	The first method
TFP1 M2	The first method	The second method
TFP2 M1	The second method	The first method
TFP2 M2	The second method	The second method

Table 6: Names of the Combinations

It is assumed that $A_1 = 0$ and $K_1 = 1$. Based on these assumptions, it was possible to construct timeseries for A_t, K_t, I_t, Y_t and C_t . K_{t+1} can be generated using the solution of the model. Then Y_t can be calculated using the production function. I_t can be generated by rearranging the capital accumulation equation. Finally, C_t can be recovered as the residual difference between Y_t and I_t .

2.3 Evaluating the Model's Performance

Before the evaluation, logs had to be taken and detrending had to be conducted. Afterwards the ratios, used to evaluate the performance of the given models, were calculated.

2.3.1 How to Handle the Data

First, the decision had to be made if generating 200 values for each variable (I) was sufficient, or if creating 300 and deducting the first 100 in order to control for initial conditions that might create a bias (II) was necessary.

In order to make a confident decision, the analysis of standard deviations was created for both ways and the results were compared. It became clear that the second approach was favorable since all standard deviations came closer to the observed values.

2.3.2 Standard Deviation

The recovered standard deviation yields the first interesting results. No calibration method was able to match every variable's standard deviation perfectly, but some of them were able to meet the observed values fairly well.

An example constitutes TFP 1 M 2 and its standard deviation for capital. While all other approaches failed to deliver the observed volatility, this model did a remarkably good job. TFP 1 M 1 got very close in the standard deviation of GDP, outperforming all other models.

It is noticeable that no model was able to fully replicate the higher volatility observed in investment or consumption to their full extend.

variable	Obs.	TFP 1 M 1	TFP 1 M 2	TFP 2 M 1	TFP 2 M 2
Y	0.0153	0.0158	0.0162	0.0118	0.0127
C	0.0123	0.0021	0.0046	0.0017	0.0034
K	0.0049	0.0021	0.0047	0.0017	0.0034
I	0.0343	0.0221	0.0182	0.0164	0.0143

Table 7: Standard Deviation Across all Variables

The afterwards computed relative standard deviation yields the result that all approaches generated measures for capital and consumption that are less volatile than GDP but very close to one another. Investment is more volatile than GDP. This partly matches the observations and modern theory, although the values are not as big as observed.

variable	Obs.	TFP 1 M 1	TFP 1 M 2	TFP 2 M 1	TFP 2 M 2
Y	1.0000	1.0000	1.0000	1.0000	1.0000
C	0.8002	0.1315	0.2874	0.1482	0.2682
K	0.3222	0.1299	0.2882	0.1469	0.2682
I	2.2385	1.3914	1.1228	1.3861	1.1246

Table 8: Relative Standard Deviation Across all Variables

2.3.3 Correlation with GDP

Concerning the correlation, TFP 1 M 2 and TFP 2 M 2 again came closest to the observed values for capital and consumption, but they still created values with a far lower correlation than actually observed.

The correlation of investment with GDP was overestimated by all four models by nearly the same amount. The correlation of consumption and capital with GDP was underestimated by all four approaches.

Variable	Obs.	TFP 1 M 1	TFP 1 M 2	TFP 2 M 1	TFP 2 M 2
Y	1.0000	1.0000	1.0000	1.0000	1.0000
C	0.7454	0.303	0.4103	0.2993	0.4098
K	0.3086	0.0415	0.1348	0.0651	0.1144
I	0.8601	0.9993	0.9995	0.9987	0.9995

Table 9: Correlation with GDP Across all Variables

2.3.4 Autocorrelation

In this section, the recovered autocorrelations will be analyzed. The tables are organized after the variable they are covering in order to provide an easy way to compare the displayed results.

First, the autocorrelation of GDP will be displayed:

The approaches TFP 1 M 1 and TFP 1 M 2 came closest to the observed autocorrelation but were still producing values that are a bit too large. TFP 2 M 1 and TFP 2 M 2 first underestimated autocorrelation, but came closer to observation with increasing lag. Overall, the autocorrelation of output was simulated quite realistically.

Lag	Obs.	TFP 1 M 1	TFP 1 M 2	TFP 2 M 1	TFP 2 M 2
1	0.7789	0.7728	0.7891	0.6071	0.6565
2	0.5514	0.5783	0.6232	0.3849	0.4252
3	0.3275	0.3726	0.4463	0.2492	0.2105
4	0.0967	0.1777	0.2474	0.1109	0.0895
5	-0.0611	0.0513	0.0401	-0.0123	-0.0579

Table 10: Autocorrelation of GDP

Autocorrelation of consumption was highly overestimated by all four models.

Lag	Obs.	TFP 1 M 1	TFP 1 M 2	TFP 2 M 1	TFP 2 M 2
1	0.5926	0.9431	0.9516	0.9147	0.9483
2	0.3981	0.8336	0.8393	0.7944	0.8363
3	0.2782	0.6844	0.6777	0.6536	0.6847
4	-0.0167	0.5142	0.4803	0.5028	0.5161
5	-0.1289	0.3373	0.2653	0.3529	0.3434

Table 11: Autocorrelation of Consumption

The autocorrelation of capital was overestimated by all approaches. With increasing lag, the simulated values converge to those observed.

Lag	Obs.	TFP 1 M 1	TFP 1 M 2	TFP 2 M 1	TFP 2 M 2
1	0.7196	0.9444	0.9524	0.9164	0.9474
2	0.4721	0.8342	0.8406	0.7979	0.8365
3	0.3306	0.6879	0.6802	0.6594	0.6884
4	0.3065	0.5201	0.4855	0.5115	0.5217
5	0.2829	0.3443	0.2723	0.3657	0.3486

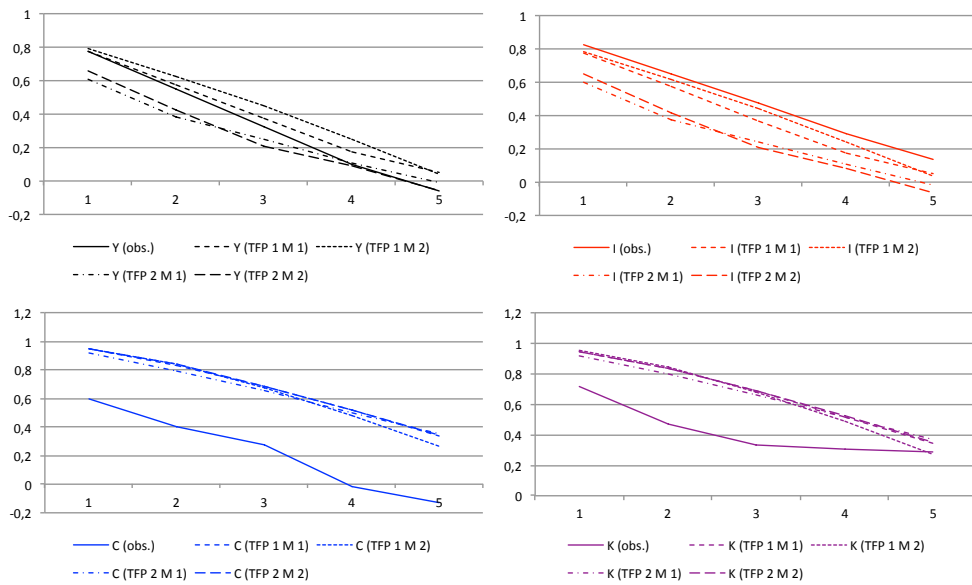
Table 12: Autocorrelation of Capital

The autocorrelation of investment was underestimated by all four approaches with TFP 1 being closer to the observation than TFP 2.

Lag	Obs.	TFP 1 M 1	TFP 1 M 2	TFP 2 M 1	TFP 2 M 2
1	0.8245	0.7709	0.7861	0.6014	0.6516
2	0.6465	0.5753	0.6193	0.3761	0.4184
3	0.4713	0.3702	0.4424	0.2411	0.2033
4	0.2897	0.1767	0.2436	0.1046	0.0832
5	0.1364	0.0511	0.0363	-0.0192	-0.0639

Table 13: Autocorrelation of Investment

Plotting the variables allows for an even more convenient comparison:



2.3.5 Lead and Lag Analysis

The crosscorrelations of all variables with GDP over time were analyzed through implementing a lag in either I_t , K_t or C_t . In order to provide an easy way to compare the variables, they are organized in tables covering each variable individually as well.

While it was observed that consumption is neither a leading or lagging indicator, all four approaches generated a lagging consumption. For the first method to recover TFP, the correlation with current GDP is greatest for C_{t+4} . For the second method it is one period closer to reality with C_{t+3} .

Lag	Obs.	TFP 1 M 1	TFP 1 M 2	TFP 2 M 1	TFP 2 M 2
t-5	-0.0029	-0.5522	-0.6101	-0.4065	-0.4194
t-4	0.1415	-0.4807	-0.4892	-0.3491	-0.3582
t-3	0.3582	-0.3582	-0.3157	-0.2681	-0.2589
t-2	0.4621	-0.1852	-0.1082	-0.1385	-0.0976
t-1	0.6063	0.0328	0.1281	0.0286	0.1149
t	0.7457	0.3030	0.4103	0.2993	0.4098
t+1	0.4863	0.4891	0.6043	0.4404	0.5762
t+2	0.3224	0.6212	0.7316	0.5061	0.6577
t+3	0.1709	0.6789	0.7992	0.5435	0.6658
t+4	-0.0222	0.6842	0.8012	0.5257	0.6369
t+5	-0.1102	0.6578	0.7450	0.4842	0.5658

Table 14: Crosscorrelation of Consumption

As observed, the model generated a capital stock that is a lagging variable. For the first method to uncover TFP, the highest correlation is observed for K_{t+5} . The second method to uncover TFP yields a capital that has the highest correlation with current GDP for K_{t+4} .

Lag	Obs.	TFP 1 M 1	TFP 1 M 2	TFP 2 M 1	TFP 2 M 2
t-5	-0.2111	-0.5923	-0.6757	-0.4237	-0.4373
t-4	-0.1443	-0.5522	-0.6086	-0.3937	-0.4194
t-3	-0.0479	-0.4778	-0.4818	-0.3435	-0.3515
t-2	0.0804	-0.3536	-0.3114	-0.2417	-0.2474
t-1	0.2437	-0.1789	-0.1064	-0.1213	-0.0928
t	0.4071	0.0415	0.1348	0.0651	0.1144
t+1	0.5338	0.2993	0.4110	0.3095	0.4095
t+2	0.6087	0.4915	0.6009	0.4326	0.5749
t+3	0.6324	0.6089	0.7295	0.5057	0.6552
t+4	0.6088	0.6687	0.7932	0.5221	0.6637
t+5	0.5545	0.6762	0.7967	0.5086	0.6354

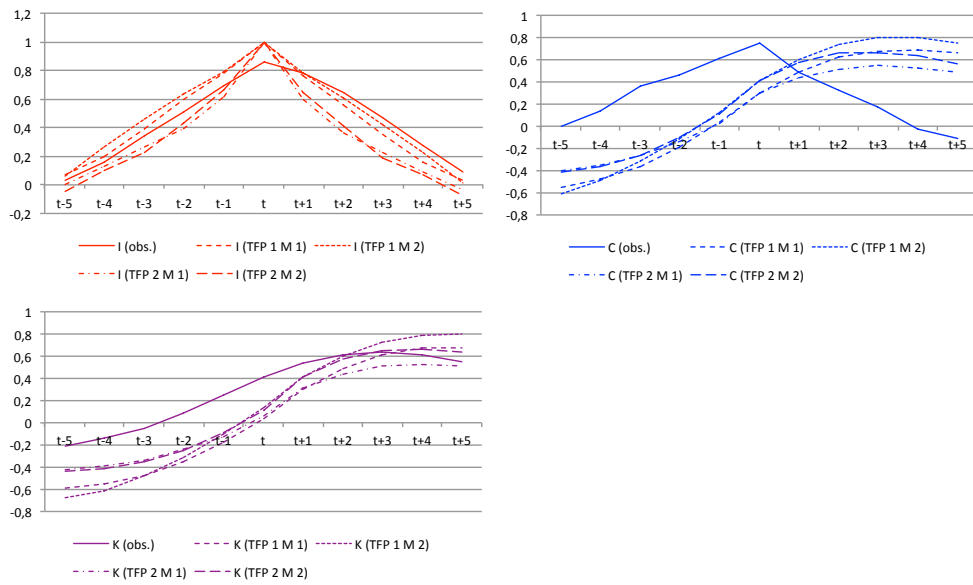
Table 15: Crosscorrelation of Capital

For investment, the findings are that, after considering the deviations of the correlations from the observed ones, the change in crosscorrelation generated by the model follows the changes in crosscorrelation that were observed. That is, having a coincident investment in capital (or fixed assets).

Lag	Obs.	TFP 1 M 1	TFP 1 M 2	TFP 2 M 1	TFP 2 M 2
t-5	0.0317	0.0742	0.0621	0.0023	-0.0446
t-4	0.1632	0.1994	0.2681	0.1252	0.1027
t-3	0.3420	0.3911	0.4638	0.2614	0.2220
t-2	0.5083	0.5919	0.6358	0.3931	0.4341
t-1	0.6977	0.7801	0.7955	0.6120	0.6613
t	0.8607	0.9993	0.9995	0.9987	0.9995
t+1	0.7811	0.7623	0.7786	0.5945	0.6459
t+2	0.6478	0.5605	0.6059	0.3662	0.4088
t+3	0.4722	0.3506	0.4242	0.2277	0.1912
t+4	0.2766	0.1542	0.2223	0.0893	0.0695
t+5	0.0889	0.0276	0.0140	-0.0343	-0.0774

Table 16: Crosscorrelation of Investment

It is again possible to plot the results in graphs in order to simplify the comparison between observed values and the generated ones:



2.4 Conclusion

After calculating, comparing and interpreting the statistics of the model-generated data, a conclusion about the validity of the simple stochastic RBC model can be drawn.

It is to say that the standard deviation and relative standard deviation were matching the basic findings in the data but not to its full extent. This essentially means that the model generated values with the characteristics $\sigma_K \approx \sigma_C < \sigma_Y < \sigma_I$.

The model was also able to replicate the very basic findings in terms of correlation ($\rho_{K,Y} < \rho_{C,Y} < \rho_{I,Y}$) as well. But in terms of absolute values there is still a big gap between the observed values and the generated ones.

In terms of autocorrelation, only GDP and investment for the TFP 1 method was measured appropriately. All other variables were either over- or underestimated.

Concerning the lead and lag analysis, the model yielded results for consumption that do not coincide with the previous findings and modern business cycle theory.

The different approaches to recover TFP did not yield great differences in the statistics. The first method to recover TFP actually performed marginally better than the second. The small difference in results comes from both parameters being very close to each other.

These findings favor the conclusion, that the used simple stochastic RBC model might be able to replicate the very basic statistics but does not describe reality to its full extent.

Evidently, there is room for improvement of the model. One could for example change the production function. Taking labor into account should improve the quality tremendously. It would also generate an additional variable to evaluate the model's performance. Furthermore, incorporating positive but decreasing marginal products with respect to capital and labor should yield better results. The nature of the shocks could be changed as well. A normally distributed shock would allow for rare, large outliers that appear in reality (i.e. the oil price shock) and thus describe the nature of the shocks in a more realistic way.

3 The RBC Model with Endogenous Labor

This sections covers the solution and evaluation of an RBC model with endogenous labor. The model is defined as:

$$\max_{(C_t, I_t, K_t, N_t)_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t [\log C_t - \frac{N_t^2}{2}]$$

s.t.

$$C_t + I_t = \exp(A_t) K_t^{0.36} N_t^{0.64}$$

$$K_{t+1} = (1 - \delta) K_t + I_t$$

$$K_0 \quad \text{given}$$

This creates a consumption-leisure tradeoff. There are two ways to solve maximization problems like this. The first approach is to solve the model by hand, the second is to build a computer model, that solves the equations by itself.

3.1 Solving the Model by Hand

Since this approach was not used to create the data that will be displayed later on, this section is dedicated to show the general procedure in a shortened manner. The approach is mainly based on a guide published by John H. Cochrane¹⁵. The first two steps are required to solve the model in Dynare as well.

3.1.1 Combining the Constraints

$$\begin{aligned} I_t &= K_{t+1} - (1 - \delta) K_t \\ Y_t = I_t + C_t &= K_{t+1} - (1 - \delta) K_t + C_t \end{aligned}$$

In order to build the Lagrangian, the constraint needs to be equal to zero.

$$0 \stackrel{!}{=} K_{t+1} - (1 - \delta) K_t + C_t - \exp(A_t) K_t^{0.36} N_t^{0.64}$$

3.1.2 Setting up the Lagrangian and Deriving the FOCs

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t [\log C_t - \frac{N_t^2}{2}] - \lambda_t \beta^t [K_{t+1} - (1 - \delta) K_t + C_t - \exp(A_t) K_t^{0.36} N_t^{0.64}]$$

Now, partial derivatives with respect to C_t , K_{t+1} , N_t and λ_t have to be calculated:

¹⁵John H. Cochrane, Solving real business cycle models by solving systems of first order conditions, April 8, 2001, available at <http://faculty.chicagobooth.edu/john.cochrane/research/papers/kpr2a.pdf>

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t \frac{1}{C_t} - \lambda_t \beta^t \stackrel{!}{=} 0$$

$$\frac{1}{C_t} = \lambda_t \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = -\lambda_t \beta^t - \beta^{t+1} E_t [\lambda_{t+1} (-(1-\delta) - 0.36 \exp(A_{t+1}) K_{t+1}^{-0.64} N_{t+1}^{0.64})] \stackrel{!}{=} 0$$

$$\frac{1}{C_t} = \beta E_t \left[\frac{1}{C_{t+1}} ((1-\delta) + 0.36 \exp(A_{t+1}) K_{t+1}^{-0.64} N_{t+1}^{0.64}) \right] \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = \beta^t (-N_t) - \lambda_t \beta^t (-0.64 \exp(A_t) K_t^{0.36} N_t^{-0.36}) \stackrel{!}{=} 0$$

$$N_t = 0.64 \frac{1}{C_t} \exp(A_t) K_t^{0.36} N_t^{-0.36} \quad (3)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = -\beta^t [K_{t+1} - (1-\delta)K_t + C_t - \exp(A_t) K_t^{0.36} N_t^{0.64}] \stackrel{!}{=} 0$$

$$K_{t+1} = (1-\delta)K_t - C_t + \exp(A_t) K_t^{0.36} N_t^{0.64} \quad (4)$$

3.1.3 Expressing FOCs in Terms that do not grow in Steady State

This is a simple rearrangement exercise:

From(2) :

$$1 = \beta E \left[\frac{C_t}{C_{t+1}} * ((1-\delta) + 0.36 \exp(A_{t+1}) K_{t+1}^{-0.64} N_{t+1}^{0.64}) \right]$$

$$1 = \beta E \left[\frac{C_t}{C_{t+1}} * R_{t+1} \right]$$

Which defines R_{t+1} as:

$$R_{t+1} = (1-\delta) + 0.36 \exp(A_{t+1}) \left(\frac{N_{t+1}}{K_{t+1}} \right)^{0.64}$$

From(3) :

$$N_t = 0.64 \frac{\exp(A_t)}{C_t} \left(\frac{N_t}{K_t} \right)^{0.64} \frac{K_t}{N_t}$$

From(4) :

$$\frac{K_{t+1}}{K_t} = (1 - \delta) - \frac{C_t}{K_t} + \exp(A_t) \left(\frac{N_t}{K_t} \right)^{0.64}$$

3.1.4 Characterizing the Nonstochastic Steady State

Assuming that capital and consumption grow at rate $G = 1+g$ in steady state.

$$1 = \beta G^{-1} R \quad (5)$$

$$R = 0.36 \exp(A) \left(\frac{N}{K} \right)^{0.64} + (1 - \delta) \quad (6)$$

$$N = 0.64 \frac{\exp(A)}{C} \left(\frac{K}{N} \right)^{0.36} \quad (7)$$

$$G = (1 - \delta) + \exp(A) \left(\frac{N}{K} \right)^{0.64} - \frac{C}{K} \quad (8)$$

N can be referred to as the fraction of time spent working and is thus constant. The proposed typical values range from $\frac{1}{4}$ to $\frac{1}{3}$. The return equation $R = 1+r$ is not growing in steady state as well.

3.1.5 Linearizing Around the Steady State

Before being linearized, logs of the values in question had to be taken. While doing so, new notation has been introduced, which defines:

$$x = \ln(X); \tilde{x}_t = \ln(X_t); x_t = \ln(X_t) - \ln(X)$$

Afterwards, a first order Taylor expansion can be applied in order to linearize around steady state.

$$\text{from (5): } 1 = \beta E \left[\frac{C_t}{C_{t+1}} * R_{t+1} \right]$$

Taking logs and setting everything as the power of e at the same time:

$$1 = \beta E[\exp(\tilde{c}_t - \tilde{c}_{t+1}) * \exp(\tilde{r}_{t+1})]$$

Taking the Taylor expansion:

$$1 = \beta G^{-1}R + \beta G^{-1}RE(c_t - c_{t+1} + r_{t+1})$$

$$\frac{1}{\beta} = [1 + E(c_t - c_{t+1} + r_{t+1})] \frac{\beta G^{-1}R}{\beta G^{-1}R}$$

$$E(c_t + c_{t+1}) = E(r_{t+1})$$

from (6): $R_{t+1} = (1 - \delta) + 0.36 \exp(A_{t+1}) K_{t+1}^{-0.64} N_{t+1}^{0.64}$

Taking logs and setting everything as the power of e at the same time:

$$\exp(\tilde{r}_{t+1}) = (1 - \delta) + \exp(A_{t+1} - 0.64\tilde{k}_{t+1} + 0.64\tilde{n}_{t+1})$$

The Taylor approximation and simplification of its result yields:

$$Rr_{t+1} = 0.36 \exp(A) \left(\frac{N}{K}\right)^{0.64} (\exp(a_{t+1}) - 0.64k_{t+1} + 0.64n_{t+1})$$

from(7): $N_t = 0.64 \frac{\exp(A_t)}{C_t} \left(\frac{K_t}{N_t}\right)^{0.36}$

Taking logs and setting everything as the power of e at the same time:

$$\exp(\tilde{n}_t) = 0.64 \exp(\exp(\tilde{a}_t) - \tilde{c}_t) \exp(0.36\tilde{k}_t - 0.36\tilde{n}_t)$$

The Taylor approximation and simplification of its result yields:

$$(0.36 + \frac{1}{N})n_t = \exp(a_t) - c_t + 0.36k_t$$

from (8): $\frac{K_{t+1}}{K_t} = (1 - \delta) - \frac{C_t}{K_t} + \exp(A_t) \left(\frac{N_t}{K_t}\right)^{0.64}$

Taking logs and setting everything as the power of e at the same time:

$$\exp(\tilde{k}_{t+1} - \tilde{k}_t) = (1 - \delta) - \exp(\tilde{c}_t - \tilde{k}_t) + \exp(\exp(\tilde{a}_t) + 0.64\tilde{n}_t - 0.64\tilde{k}_t)$$

The Taylor approximation and simplification of its result yields:

$$Gk_{t+1} = \exp(A) \left(\frac{N}{K}\right)^{0.64} (\exp(a) + 0.64n_t) - \frac{C}{K}c_t + Rk_t$$

Summarizing the results:

$$E_t(c_{t+1}) = c_t + E_t(r_{t+1}) \tag{9}$$

$$Rr_{t+1} = (r + \delta)(\exp(a_{t+1}) + 0.64n_{t+1} - 0.64k_{t+1}) \tag{10}$$

$$\left(0.36 + \frac{1}{N}\right) n_t = \exp(a_t) - c_t + 0.36k_t \quad (11)$$

$$Gk_{t+1} = \exp(A) \left(\frac{N}{K}\right)^{0.64} (\exp(a) + 0.64n_t) - \frac{C}{K}c_t + Rk_t \quad (12)$$

Substituting (10) into (9) by dividing (10) by R yields:

$$E_t(c_{t+1}) = c_t + \frac{r + \delta}{1 + r} (\exp(a_{t+1}) + 0.64n_{t+1} - 0.64k_{t+1}) \quad (13)$$

Simplifying the constants to b-values:

$$E_t(c_{t+1}) = b_{cc}c_t + b_{ck}k_{t+1} + b_{ca}E_t a_{t+1} + b_{cn}E_t n_{t+1}$$

$$n_t = b_{nc}c_t + b_{nk}k_t + b_{na}a_t$$

$$k_{t+1} = b_{kc}c_t + b_{kk}k_t + b_{ka}a_t + b_{kn}n_t$$

This enables to organize the system using algebra matrix notation.

Directly substituting in the equation for n_t yields:

$$\begin{bmatrix} c_{t+1} \\ k_{t+1} \\ a_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & b_{ck} & b_{ca} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_{t+1} \\ k_{t+1} \\ a_{t+1} \end{bmatrix} + \begin{bmatrix} b_{cc} & 0 & 0 \\ b_{kc} & b_{kk} & b_{ka} \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix} + \begin{bmatrix} b_{cn} \\ 0 \\ 0 \end{bmatrix} [b_{nc} \quad b_{nk} \quad b_{na}] \begin{bmatrix} c_{t+1} \\ k_{t+1} \\ a_{t+1} \end{bmatrix} + \begin{bmatrix} 0 \\ b_{kn} \\ 0 \end{bmatrix} [b_{nc} \quad b_{nk} \quad b_{na}] \begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix}$$

Rearranging to have all values that affect the t+1 values on the lefthand side and everything that affects the variables in t on the righthand side:

$$\left(I - \begin{bmatrix} 0 & b_{ck} & b_{ca} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} b_{cn} \\ 0 \\ 0 \end{bmatrix} [b_{nc} \quad b_{nk} \quad b_{na}] \right) \begin{bmatrix} c_{t+1} \\ k_{t+1} \\ a_{t+1} \end{bmatrix} = \left(\begin{bmatrix} b_{cc} & 0 & 0 \\ b_{kc} & b_{kk} & b_{ka} \\ 0 & 0 & \rho \end{bmatrix} + \begin{bmatrix} 0 \\ b_{kn} \\ 0 \end{bmatrix} [b_{nc} \quad b_{nk} \quad b_{na}] \right) \begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix}$$

The matrices within the parentheses can be added up, since 3x3 matrices are the results of the multiplication. In order to provide equations that are easier to read, they can be summarized into A and B matrices.

$$A \begin{bmatrix} c_{t+1} \\ k_{t+1} \\ a_{t+1} \end{bmatrix} = B \begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix}$$

Multiplying with the inverse of A then results in a system of equations that describes future values, only depending on current values of the same variables:

$$\begin{bmatrix} c_{t+1} \\ k_{t+1} \\ a_{t+1} \end{bmatrix} = A^{-1}B \begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix}$$

Solving this yields the following equations:

$$\begin{aligned} c_{t+1} &= \left(\frac{b_{cc}}{1-b_{cn}b_{nc}} + \frac{b_{cn}b_{nk}+b_{ck}}{1-b_{cn}b_{nc}} (b_{kn}b_{nc} + b_{kc}) \right) c_t + \left((b_{kk} + b_{kn}b_{nk}) \frac{b_{cn}b_{nk}+b_{ck}}{1-b_{cn}b_{nc}} \right) k_t \\ &\quad + \left((b_{na} + b_{kn}b_{na}) \frac{b_{cn}b_{nk}+b_{ck}}{1-b_{cn}b_{nc}} + \rho \left(\frac{b_{na}b_{cn}+b_{ca}}{1-b_{cn}b_{nc}} \right) \right) exp(a_t) \\ k_{t+1} &= (b_{kc} + b_{kn}b_{nc})c_t + (b_{kk} + b_{kn}b_{nk})k_t + (b_{ka} + b_{kn}b_{na})exp(a_t) \\ A_{t+1} &= \rho \cdot exp(a_t) \end{aligned}$$

3.1.6 Campbell's Method to Solve the System

It was now possible to apply Campbell's method to the problem by first summarizing the different b coefficients to d coefficients:

$$c_{t+1} = d_{cc}c_t + d_{ck}k_t + d_{ca}exp(a_t) \quad (1)$$

$$k_{t+1} = d_{kc}c_t + d_{kk}k_t + d_{ka}exp(a_t) \quad (2)$$

$$A_{t+1} = \rho A_t + \epsilon_{t+1} \quad (3)$$

The assumption that consumption is a linear function of the state variables k_t and a_t yields:

$$c_t = \eta_{ck}k_t + \eta_{ca}exp(a_t)$$

Substituting the guess into the set of linear equations:

$$E(\eta_{ck}k_{t+1} + \eta_{ca}exp(a_{t+1})) = d_{cc}(\eta_{ck}k_t + \eta_{ca}exp(a_t)) + d_{ck}k_t + d_{ca}exp(a_t) \quad (4)$$

$$k_{t+1} = d_{kc}(\eta_{ck}k_t + \eta_{ca}exp(a_t)) + d_{kk}k_t + d_{ka}exp(a_t) \quad (5)$$

$$A_{t+1} = \rho A_t + \epsilon_{t+1} \quad (6)$$

It is now possible to simplify by combining (4) and (6) while rearranging (5):

$$\eta_{ck}k_{t+1} = (d_{cc}\eta_{ck} + d_{ck})k_t + (d_{ca} + \eta_{ca}(d_{cc} - \rho))exp(a_t)$$

$$k_{t+1} = (d_{kc}\eta_{ck} + d_{kk})k_t + (d_{kc}\eta_{ca} + d_{ka})exp(a_t)$$

Simplifying the combined equation and solving for 0:

$$[\eta_{ck}(d_{kc}\eta_{ck} + d_{kk}) - (d_{cc}\eta_{ck} + d_{ck})]k_t + [\eta_{ck}(d_{kc}\eta_{ca} + d_{ka}) - (d_{ca} + \eta_{ca}(d_{cc} - \rho))]exp(a_t) = 0$$

This must hold for every value of k and a. This implies, that each term must be zero separately. Only considering the first part of the formula and rearranging results in:

$$d_{kc}\eta_{ck}^2 + (d_{kk} - d_{cc})\eta_{ck} - d_{ck} = 0$$

This can be solved by applying the pq formula:

$$\eta_{ck1,2} = -\frac{(d_{kk} - d_{cc})}{2d_{kc}} \pm \sqrt{\left(\frac{d_{kk} - d_{cc}}{2d_{kc}}\right)^2 + \frac{d_{ck}}{d_{kc}}}$$

Since the pq formula leads to two values, it is best to try simulating the model with both of them to see which result fits best to the real data. Since the value for η_{ck} is already known, it is now possible to rearrange the second term and solve it:

$$\eta_{ca} = \frac{d_{ca} - \eta_{ck}}{\eta_{ck}d_{kc} - d_{cc} + \rho}$$

Now that the model is solved, one can simulate the model based on the two equations:

$$A_{t+1} = \rho A_t + \epsilon_{t+1}$$

$$k_{t+1} = (d_{kc}\eta_{ck} + d_{kk})k_t + (d_{kc}\eta_{ca} + d_{ka})exp(a_t)$$

values for c_t can be calculated according to:

$$c_t = \eta_{ck}k_t + \eta_{ca}exp(a_t)$$

It is however still necessary to find values for investment, which can be achieved with a linear rule for investment:

$$I_t = Y_t - C_t \Rightarrow i_t = \frac{Y}{I}y_t - \frac{C}{I}c_t$$

One thus has to find values for for the two ratios. First, the $\frac{Y}{K}$ ratio needs to be found in order to receive the $\frac{C}{K}$ ratio. This follows from rearranging (6). The result can then be used to recover the $\frac{I}{K}$ ratio, that can be used to solve for the $\frac{Y}{I}$ ratio. Afterwards, the $\frac{C}{I}$ ratio can be determined:

$$\begin{aligned}
\frac{Y}{K} &= \exp(A) N^{0.64} \frac{K^{0.36}}{K} & G &= (1 - \delta) + \exp(A) \left(\frac{N}{K}\right)^{0.64} - \frac{C}{K} & \frac{I}{K} &= \frac{Y}{K} - \frac{C}{K} \\
\frac{Y}{K} &= \exp(A) \left(\frac{N}{K}\right)^{0.64} & 1 + g &= (1 - \delta) + \exp(A) \left(\frac{N}{K}\right)^{0.64} - \frac{C}{K} & \frac{I}{K} &= \frac{r+\delta}{0.36} + (\delta + g) - \frac{r+\delta}{0.36} \\
\frac{Y}{K} &= \frac{r+\delta}{0.36} & \frac{C}{K} &= -(\delta + g) + \frac{r+\delta}{0.36} & \frac{I}{K} &= \delta + g
\end{aligned}$$

This enables to solve for the ratios needed to calculate i_t :

$$\begin{aligned}
\frac{Y}{I} &= \frac{I}{I} + \frac{C}{I} & \frac{C}{I} &= \frac{Y}{I} - 1 \\
\frac{Y}{I} - 1 &= \frac{C}{K} \frac{K}{I} & \frac{C}{I} &= \frac{r+\delta}{0.36} \cdot \frac{1}{\delta+g} - 1 \\
\frac{Y}{I} - 1 &= \left(-(\delta + g) + \frac{r+\delta}{0.36}\right) \cdot \frac{1}{\delta+g} \\
\frac{Y}{I} &= \frac{r+\delta}{0.36} \cdot \frac{1}{\delta+g}
\end{aligned}$$

3.1.7 Evaluating the Solution Derived by Hand

The outputs of the model are by definition percentage deviations from steady state and thus strongly depend on the estimated steady state growth rate.

It is assumed, that all variables grow at the same rate in steady state. Under- or overestimating the growth rate has drastic implications on the performance of the model. The constant steady state values have to be estimated as well in order to simulate the model.

All the required estimations reduce the quality and reliability of the results drastically in hindsight of the model's performance.

In order to not base the evaluation of the model on such uncertainties, it was decided to simulate the model by using Dynare¹⁶. This allowed to skip the estimation of steady state growth rates and constant values.

3.2 Solving the Model in Dynare

In order to receive a solution, the following code had to be written:

Dynare differentiates between endogenous and exogenous variables. They thus need to be defined separately:

```

var y c k i n a;
varexo e;

```

Afterwards, the parameters used by the model have to be defined and assigned to specific values. Since the calibration methods did not seem to have a great impact on the quality

¹⁶available at <http://www.dynare.org>

of the simulation, it was decided to only run the simulation for the first approach and not twice:

```

parameters alpha beta delta rho sigma;
alpha = 0.36;
beta = 0.99;
delta = 0.025;
rho = 0.971296;
sigma = 0.010987;

```

Where sigma and rho were taken from the AR(1) process from the second procedure to recover TFP. Now, the model needed to be defined by the first order conditions and market clearing conditions. It is essential to have as many equations as endogenous variables, otherwise the model can not be solved for. Dynare does the linearization itself, but not log-linearization as required for comparable statistics. The first order conditions thus had to be rewritten in terms of $\exp(x)$. This way, $\exp(x)$ was treated as the level of the variable, whereas x was the natural logarithm¹⁷. This procedure yields:

```

model;
(1/exp(c)) = beta*(1/exp(c(+1)))*((1-delta)+alpha*(exp(k)^(alpha-1))*(exp(exp(a(+1))))*exp(n(+1))^(1-alpha));
exp(n) = (1/exp(c))*(1-alpha)*(exp(k)^alpha)*(exp(exp(a)))*(exp(n)^(1-alpha));
exp(c)+exp(i) = exp(y);
exp(y) = (exp(k)^alpha)*exp(exp(a))*(exp(n)^(1-alpha));
exp(i) = exp(k)-(1-delta)*exp(k(-1));
exp(a) = rho*exp(a(-1))+e;
end;

```

Next on, the steady state guesses had to be defined. Dynare needs guesses that do not differ too much from their real value in order to simulate the model. The following guesses are rounded results that were received after running the non-log model, put into logs:

```

initval;
k = log(35);
c = log(2.6);
n = log(0.9);
a = 0;
e = 0;
end;

```

Now, the exogenous stochastic shock had to be implemented. The shock's variance

¹⁷Eric Sims, Graduate Macro Theory II: Notes on Using Dynare, University of Notre Dame, Spring 2011 available at http://www3.nd.edu/~esims1/using_dynare.pdf

needed to be defined. This follows:

```
shocks;  
var e = sigma ^ 2;  
end;
```

Finally, the simulation had to be started by writing the last lines of code:

```
steady;  
stoch_simul(irf=200, order= 1, hp_filter=1600);
```

By writing the `hp_filter=1600` command, Dynare automatically applied a Hodrick-Prescott filter to the data and presented the statistics of the cyclical components. The model could thus be evaluated immediately.

3.3 Evaluating the Model's Performance

The `irf=200` command plotted the impulse response functions of the endogenous variables. Since Dynare uses theoretical moments derived from the linear Gaussian state space representation, it does not depend on sample length¹⁸ and by that not on the amount of shocks. One shock was therefore sufficient for the model to calculate all relevant statistics. The simulated IFRs of the log-variables look as follows:

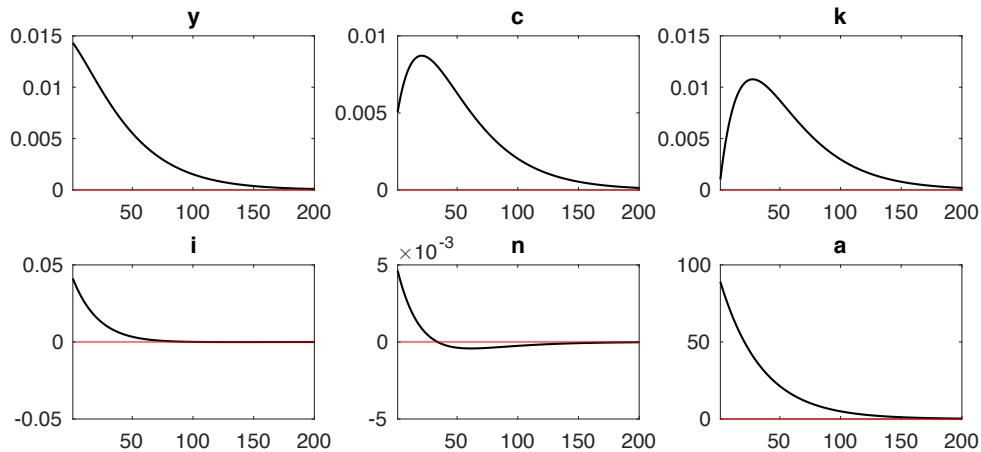


Figure 10: IRFs

¹⁸Johannes Pfeifer, University of Mannheim, <http://www.dynare.org/phpBB3/viewtopic.php?f=1&t=8167&p=24205&hilit=number+of+shocks#p24205> , reviewed on 11.6.2016

3.3.1 Standard Deviation and Correlation with GDP

In terms of standard deviation as well as relative standard deviation, it can be seen that the model matched the data pretty well. Only the standard deviation of consumption was underestimated, while the standard deviation for investment was overestimated by noticeable amounts.

	Std. (model)	Std. (obs.)	Rel. Std. (model)	Rel. Std. (obs.)
Y	0.0187	0.0153	1.0000	1.0000
C	0.0072	0.0123	0.3851	0.8002
K	0.0048	0.0049	0.2567	0.3222
I	0.0537	0.0343	2.8717	2.2385
N	0.0061	0.0048	0.3262	0.3165

Table 17: Simulated versus Observed Standard Deviation

Just like in the simple model, correlation between investment and GDP was overestimated. This model overestimated all correlations. Nevertheless the differences are smaller than in the simple model. Labor constitutes an exception as it has a highly overestimated correlation with GDP.

	Corr. (model)	Corr. (obs.)
C	0.9449	0.7454
K	0.3496	0.3086
I	0.9919	0.8601
N	0.9811	0.4585

Table 18: Simulated versus Observed Correlation with GDP

3.3.2 Autocorrelation

For output, the model generated values that initially have a lower persistence which is then slower decreasing than observed. The persistence of consumption and capital was overestimated. For investment and employment, persistence was underestimated.

		1	2	3	4	5
model	Y	0.7242	0.4886	0.2915	0.1305	0.0028
	C	0.7817	0.5801	0.3987	0.2395	0.1033
	K	0.9606	0.8651	0.7326	0.5788	0.4165
	I	0.7146	0.4732	0.2735	0.1122	-0.0142
	N	0.7127	0.4702	0.2699	0.1086	-0.0175
obs.	Y	0.7789	0.5514	0.3275	0.0967	-0.0610
	C	0.5926	0.3981	0.278	-0.0167	-0.1289
	K	0.7196	0.4721	0.3306	0.3065	0.2829
	I	0.8245	0.6465	0.4713	0.2897	0.1364
	N	0.7753	0.6369	0.4522	0.2843	0.1470

Table 19: Simulated versus Observed Autocorrelation

Except for capital and consumption, the differences are not of great magnitude. It is easier to compare the autocorrelations by plotting them:

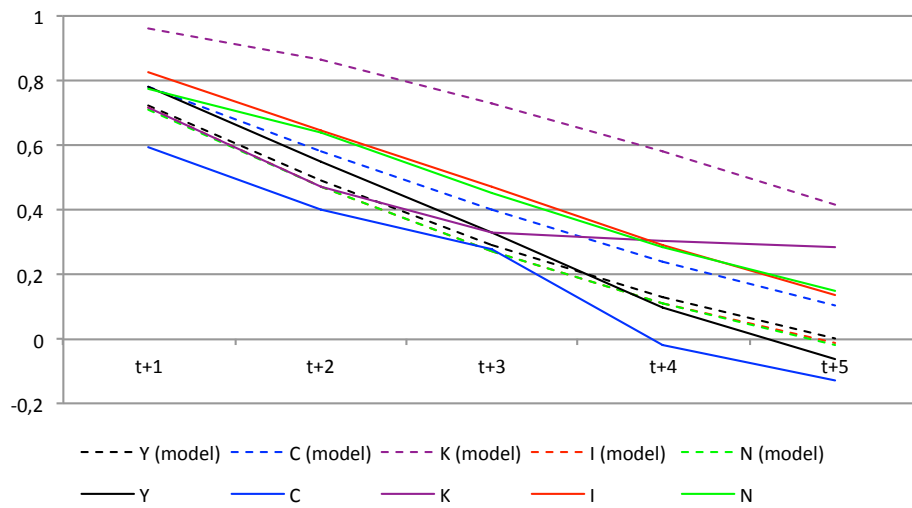


Figure 11: Simulated versus Observed Autocorrelation

3.3.3 Lead and Lag Analysis

Although there are some deviations from the recovered data, there is a huge improvement in the quality of the data regarding lags:

		t-5	t-4	t-3	t-2	t-1	t	t+1	t+2	t+3	t+4	t+5
model	C	-0.1574	-0.0299	0.1392	0.3550	0.6222	0.9449	0.7812	0.6237	0.4765	0.3425	0.2234
	K	-0.4569	-0.3929	-0.2880	-0.1344	0.0755	0.3496	0.5303	0.6344	0.6773	0.6728	0.6329
	I	0.0648	0.1889	0.3420	0.5260	0.7425	0.9919	0.6808	0.4218	0.2111	0.0445	-0.0829
	N	0.0973	0.2187	0.3665	0.5424	0.7472	0.9811	0.6533	0.3836	0.1671	-0.0015	-0.1279
obs.	C	-0.0029	0.1415	0.3582	0.4620	0.6063	0.7454	0.4863	0.3224	0.1709	-0.0222	-0.1102
	K	-0.2111	-0.1443	-0.0474	0.0804	0.2437	0.3086	0.5338	0.6087	0.6324	0.6088	0.5545
	I	0.0317	0.1632	0.3420	0.5083	0.6977	0.8601	0.7811	0.6478	0.4720	0.2766	0.0889
	N	-0.1744	-0.1116	0.0282	0.1853	0.3476	0.4585	0.5227	0.5329	0.4690	0.3569	0.1102

Table 20: Simulated versus Observed Crosscorrelation

Capital is still a lagging variable, since it has its highest correlation with current output in $t+3$, just as observed in the real data.

N was simulated not as lagging variable but as coincident which constitutes a shortcoming of the model.

Consumption is now coincident, just as observed.

Again, these findings can be plotted for easier comparison:

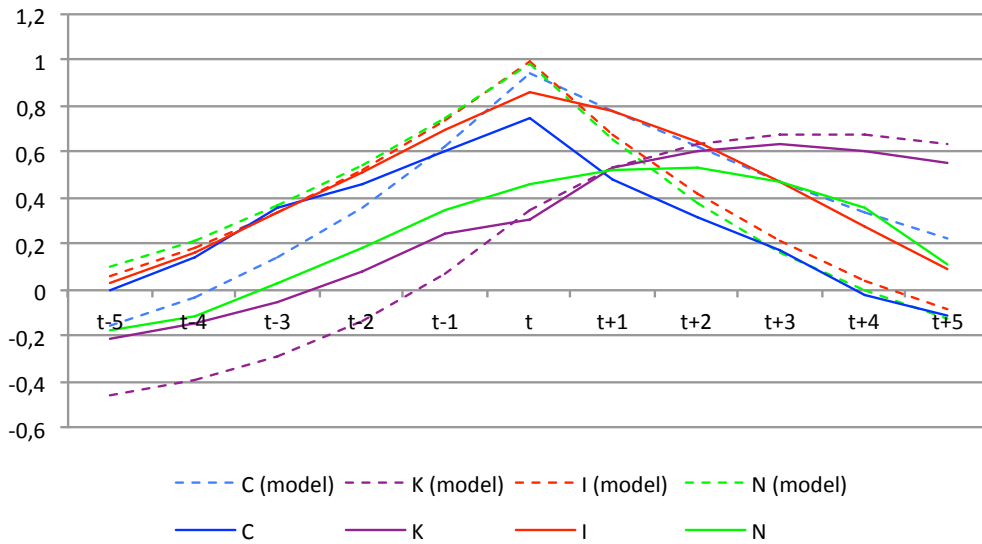


Figure 12: Simulated versus Observed Crosscorrelation

3.4 Conclusion

Introducing variable labor and positive but decreasing marginal returns of capital and labor improved the quality of the model drastically. Overall, it came closer to the observed values in all statistical measures compared to the simple stochastic model.

But there is still room for improvement. As it was explained earlier, labor is usually a lagging variable because of investment in employees by the firms (e.g. on-the-job training). On the side of the employee, the time consuming search for a job has to be conducted after the consumption-leisure decision has been made. This very basic fact was not replicated by the model. Thus, including frictions in the labor market should create a lagging employment.

Investment goes to physical capital only, not to any other sources. Capital has an even lower impact on output compared to the simple stochastic model because it now does not have constant returns to scale anymore. Introducing a model that allows for investment in human capital would increase the impact of today's investment on tomorrow's output. This could help to make investment a leading variable.

Remembering what mainly caused the shock to the Japanese economy during the financial crisis in 2008, one could think of another external shock i.e. to the foreign business climate¹⁹. It is open to discuss, whether a model with such an addition would still fall under the definition of a real business cycle model.

Since only introducing variable labor and positive but decreasing returns of capital and labor already improved the quality of the model tremendously, it is likely that including the two modifications mentioned before the additional shock will be sufficient to describe reality very closely.

It is to see whether this guess holds true or if additional modifications have to be made that would further increase the complexity of the model.

Declaration of Authorship:

I hereby certify that the work presented is entirely based on my own independent research except stated otherwise.

Sebastian Kühnl
30.6.2016

¹⁹Bernd Lucke, *Studies in Contemporary Economics, Theorie und Empirie realer Konjunkturzyklen*, Physica-Verlag Heidelberg 1998, page 359